A generalization of Vdovichenko's method for Ising models on torus graphs

1999 J. Phys. A: Math. Gen. 32 6915

(http://iopscience.iop.org/0305-4470/32/40/302)

J. Phys. A: Math. Gen. **32** (1999) 6915–6925. Printed in the UK PII: S0305-4470(99)02675-X

A generalization of Vdovichenko's method for Ising models on torus graphs

Makoto Idzumi

Department of Mathematics, Shimane University, Matsue 690-8504, Japan

E-mail: idzumi@edu.shimane-u.ac.jp

Received 12 March 1999, in final form 30 June 1999

Abstract. A new method of counting polygons on a graph embedded into a torus is described. It is based on Vdovichenko's random walk approach to planar Ising models which uses a connected loop expansion. The Ising partition function of the graph is expressed as a single determinant with some nonanalytic operation. The most generalized version of Vdovichenko's formula for counting polygons on a plane graph is also presented.

1. Introduction

This paper concerns an exact combinatorial method of counting polygons on an arbitrary *finite graph* embedded into a *torus* to evaluate the Ising partition function. Our main interest is in taking the topology of the graph (in other words boundary conditions) correctly into account. We shall propose a new algorithm which enables one to write down the partition function as a single determinant with some nonanalytic operation (see equations (7) and (8) in section 4).

Historically, Kaufman [1] first obtained an explicit expression of the Ising partition function for the finite-size toroidal square lattice (finite lattice wrapped on a torus) using her algebraic method. In 1952, Kac and Ward [2] proposed a purely combinatorial approach to the Ising problem and gave a determinant formula for the partition function which was exact in the large lattice-size limit (that is, which could rederive Onsager's result [3]); but they neglected 'boundary effects'. Subsequent work by Potts and Ward [4] described a recipe for treating the toroidal boundary conditions in the framework of Kac and Ward and successfully rederived Kaufman's result; they wrote down the partition function as a linear combination of four square roots of determinants. In the early 1960s, Kasteleyn [5] and Fisher [6] independently studied the statistics of dimers on lattices (and graphs) and discovered a relation between the dimer problem and the Ising problem. (They established the Pfaffian method for the Ising problem; the connection between the Ising problem and Pfaffians was first discovered by Hurst and Green [7] through the use of fermions.) Kasteleyn used the device of Potts and Ward to treat the toroidal boundary condition. (Furthermore, in the course of the study he (or they) noticed that the device of Potts and Ward can be generalized for graphs drawn on surfaces of genus greater than one; for a graph drawn on a surface of genus g one needs 4^g determinants to express the partition function—so commented the introduction in [8].)

Yet anothor combinatorial approach was proposed by Vdovichenko [9]. The basic idea was to use (or establish) the connected loop expansion for the counting problem on planar square lattices. He succeeded in rederiving Onsager's solution [3] to the model on the square lattice in the thermodynamic limit; but he was not interested in boundary conditions.

The purpose of this paper is to present a new method, based on Vdovichenko's, of counting polygons on a *finite graph embedded into a torus*. We shall introduce two commuting variables, a and b, which correspond to canonical basis cycles on a torus, in the definition of Vdovichenko's random walk transition matrix; then the exponential of the connected loop expansion gives a certain loop expansion for a graph on a covering surface (in fact, a plane) which is induced by the original graph drawn on the torus; finally, if we do a certain operation regarding the variables a and b on the loop expansion then we obtain the exact partition function. This is an outline of our method which will be described in section 4. Section 3 is devoted to exposing the most generalized version of Vdovichenko's method for *planar graphs* because the author has not seen it in the literature.

2. Mathematical formulation of the problem

Let us begin with the precise definition of the Ising model on a finite graph. Some topological concepts on graphs are also introduced.

2.1. The Ising model on a finite graph

A graph (or digraph) G is a triple of a set $V(G)$ of vertices, a set $E(G)$ of edges, and an assignment ϕ_G of an ordered pair $(I_G(e), T_G(e))$ of elements of $V(G)$ to each edge $e \in E(G)$. $I_G(e)$ and $T_G(e)$ are called the initial vertex and the terminal vertex of the edge e, respectively; they are called endpoints of e.

Let $G = (V(G), E(G), \phi_G)$ be a finite graph (that is, let both $V(G)$ and $E(G)$ be finite). Let $\mathcal{C}_G$ denote the set of all functions $\sigma : V(G) \to \{-1, 1\}, \alpha \mapsto \sigma_\alpha$ (an element σ of $\mathcal{C}_G$ is called a spin configuration). Let $\boldsymbol{K} : E(G) \to \boldsymbol{C}, e \mapsto K_e$ be a function. The Ising model on the graph G associated with the interaction $\boldsymbol{K}$ is defined by the partition function

$$Z = Z(\boldsymbol{K}; G) = \sum_{\sigma\in\mathcal{C}_G} \exp\left(\sum_{e\in E(G)} K_e \sigma_{I_G(e)}\sigma_{T_G(e)}\right). \tag{1}$$

Let us introduce a few more definitions. A graph $H = (V(H), E(H), \phi_H)$ is called a subgraph of G if $V(H) \subset V(G)$, $E(H) \subset E(G)$ and $\phi_H = \phi_G|_{E(H)}$ hold. For each vertex $\alpha \in G$ the number

$$\deg_G \alpha = \#\{e \in E(G) | I_G(e) = \alpha\} + \#\{e \in E(G) | T_G(e) = \alpha\}$$

is called the degree of α in G. By a *polygon* P on G we shall mean a subgraph of G in which every vertex (of P) has even degree $\geqslant 2$ in P (hence, a polygon itself is an Eulerian graph, and so it seems that an *Eulerian subgraph* may be a better naming; but we follow tradition). Let $\mathcal{P}(G)$ denote the set of all polygons on G.

Substituting $\mathrm{e}^{K\sigma\sigma} = \cosh K(1 + \sigma\sigma \tanh K)$ we can rewrite (1) as

$$\begin{aligned} Z &= 2^{\#V(G)} \left\{ \prod_{e\in E(G)} (1 - x_e^2)^{-\frac{1}{2}} \right\} \cdot S \\ S &= S(\{x_e | e \in E(G)\}) = \sum_{P\in\mathcal{P}(G)} \prod_{e\in E(P)} x_e \end{aligned} \tag{2}$$

where $x_e = \tanh K_e$. (In order that this conversion makes sense we must assume each $K_e \notin \{\mathrm{i}\pi(m + \frac{1}{2}) | m \in \mathbb{Z}\}$.) From now on, we shall forget what the x_e were and regard S as a formal power series (rather than a polynomial) in x_e. Thus, the original Ising problem (1) has been converted to a combinatorial enumeration problem (2).

2.2. Facts about topology of graphs

A finite graph can be regarded as a topological space in the standard manner: that is, each edge is assumed to be homeomorphic to the real interval $[0, 1]$ where the initial (respectively, terminal) vertex corresponds to the endpoint zero (respectively, one) of the interval, and so on. A continuous mapping of a graph into a topological space is called an embedding if the graph and its image is homeomorphic.

It is known that any finite graph can be embedded smoothly into some compact orientable surface (i.e., any graph can be drawn on the surface in such a way that the images of edges are smooth arcs and do not have intersections). Only smooth embeddings will be considered in this paper.

A graph which can be embedded into a surface of genus g but not into one of genus $g-1$ is called a graph of genus g. A graph of genus zero is called planar (since if a graph is embeddable into a sphere then it is embeddable into a plane and vice versa). For an embedding of a graph into a plane the image of the graph in the plane (the figure of the graph drawn on the plane) is called a plane graph. Analogously for an embedding of a graph into a torus the image of the graph in the torus (the figure of the graph drawn on the torus) is called a torus graph.

In the next section we consider a plane graph and give a formula for the partition function (2). In section 4 we consider a torus graph.

3. The partition function for a plane graph

Although the main purpose of this paper is to describe a new algorithm for a torus graph, we consider here a plane graph. We present a generalized version of Vdovichenko's formula for the partition function for a plane graph. The result might be known to the specialists; however, the author has not found it in the literature.

3.1. A generalized version of Vdovichenko's random walk representation and a determinant formula

Let G be a planar finite graph and let f be a smooth embedding of G into a plane. From now on we use M to denote $\#E(G)$.

Following Vdovichenko [9], we consider a walk on the plane graph $f(G)$. Note that on each edge $e \in E(G)$, and therefore on $f(e)$ as well, an orientation (from the initial vertex to the terminal) is naturally defined by the homeomorphism $e \simeq [0, 1]$ which was used to introduce a topology on the graph G. Let us introduce an index set $I = \{e | e \in E(G)\} \sqcup \{-e | e \in E(G)\}$ consisting of $2M$ objects: the index $e \in I$ denotes a move along the edge $f(e)$ in the + direction and the index $-e \in I$ denotes a move along $f(e)$ in the $-$ direction. Let $b \in I$: by $-b$ we shall mean $-e$ if $b = e$; e if $b = -e$. For $b, b' \in I$ let us say that 'a transition $b' \to b$ is *allowed*' if the terminal point of b' coincides with the initial point of b and $b \neq -b'$. For such a pair $b, b' \in I$ with $b' \to b$ allowed we define the transition 'probability' $W_{bb'}$ from the 'state' b' to b as follows:

$$W_{bb'} = x_{b'} \cdot \exp\left(\frac{\mathrm{i}}{2}\left(\varphi_{bb'} + \int_{b'} k \,\mathrm{d}s\right)\right) \tag{3}$$

where $\varphi_{bb'}$ is the angle of b relative to b' at the terminal point of b' (see figure 1). Note that here symbols $b \in I$ are used to denote the corresponding oriented arcs (which are edges of the plane graph $f(G)$) in the plane, k is the geodesic curvature of the arc b', s is the arc length parameter for b', and $x_{b'} = x_e$ if $b' = e$ or $-e$ (where $e \in E(G)$). For pairs b, b' for which $b' \to b$ are not allowed we define $W_{bb'} = 0$.

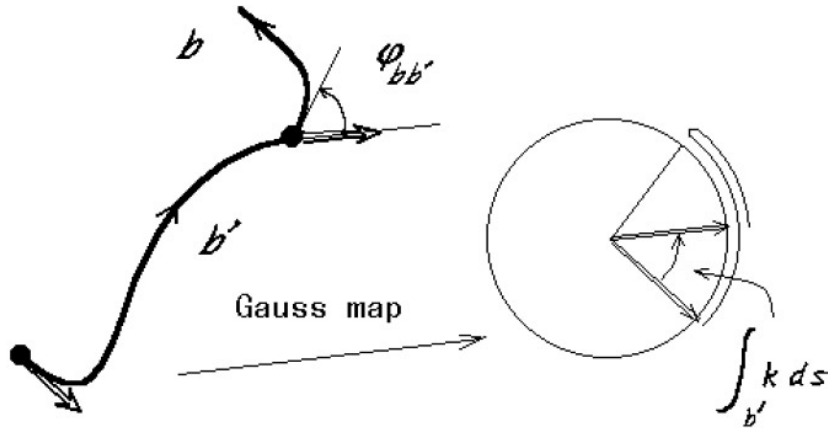

Figure 1. $\varphi_{bb'}$ and $\int_{b'} k\,\mathrm{d}s$.

Now a straightforward generalization of Vdovichenko's formula is stated.

Theorem. *Let G be a planar finite graph and let f be a smooth embedding of G into a plane. Let W be a $2M \times 2M$ matrix whose (b, b')-elements are $W_{bb'}$. Then the following formula holds for (2)*:

$$S(\{x_e | e \in E(G)\}) = \exp\left(- \sum_{r=1}^{\infty} \frac{1}{2r} \operatorname{trace}(W^r) \right). \tag{4}$$

Vdovichenko [9] originally obtained a formula of this form for the square lattice in the plane (neglecting boundary conditions). It is clear that his argument can be extended for any plane graph whose edges are not necessarily straight segments.

In fact, $\varphi_{bb'} + \int_{b'} k\,\mathrm{d}s$ in our $W_{bb'}$ measures the change in direction when a walk moves along the arc b' and turns to b at the terminal point of b' (see figure 1); hence, the argument of the exponential in (4) correctly sums up the contributions from connected loops (the planarity of $f(G)$ is essential here). Therefore, the exponential in (4) yields a summation over 'superpositions of connected loops' (superpositions, for short). Contributions of the superpositions having 'repeated edges' cancel in the sum; so what remains is a summation over superpositions having no repeated edges which correspond to polygons; but, again, cancellation occurs, and we obtain correct counting of polygons. (To see this, consider a polygon P having p intersections $1, \ldots, p$ whose degree in P are $2m_j$, $j = 1, \ldots, p$, respectively (m_j are integers $\geqslant 2$); then the number of superpositions which correspond to P (in other words, the number of ways of decomposing the polygon P into connected loops) is $(2m_1 - 1)!! \times \cdots \times (2m_p - 1)!!$. An intersection vertex of degree $2m$ has $2m$ incident edges; the number of ways of decomposing these $2m$ incident edges into m routes passing through the vertex is $(2m - 1)!!$ (this is odd whatever m is); $((2m-1)!!+1)/2$ ways have even transversal intersections in a neighbourhood of the vertex, and the remaining $((2m-1)!!-1)/2$ ways have odd intersections (the difference between the two numbers is 1). Such a local property ensures that in the summation over the $(2m_1 - 1)!! \times \cdots \times (2m_p - 1)!!$ superpositions $(2m_1 - 1)!! \times \cdots \times (2m_p - 1)!! - 1$ terms cancel out and what remains is just a single term which represents the correct weight for the polygon P.)

As a corollary we obtain

$$S(\{x_e | e \in E(G)\}) = [\det(I - W)]^{1/2}. \tag{5}$$

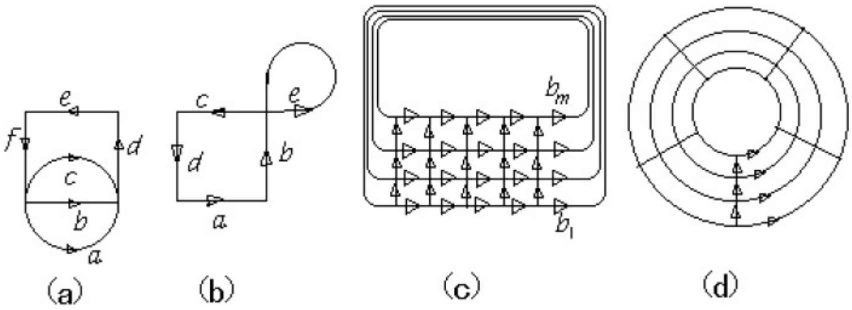

Figure 2. Planar graphs embedded into the plane: (*a*) a graph with parallel edges; (*b*) a graph with a closed edge; (*c*) the $n \times m$ square lattice with cyclic boundary condition in the horizontal direction; (*d*) anothor embedding of the same lattice. Arrows show reference orientations assigned on edges.

For a proof it is sufficient to recall the fact that any matrix can be transformed into an upper-triangle matrix by some similarity transformation.

Remark. *The parametrization of the matrix W is not unique. For example, more symmetric parametrization is possible*:

$$W_{bb'} = (x_b x_{b'})^{1/2} \cdot \exp\left(\frac{\mathrm{i}}{2}\left(\varphi_{bb'} + \frac{1}{2}\int_b k\,\mathrm{d}s + \frac{1}{2}\int_{b'} k\,\mathrm{d}s\right)\right).$$

This parametrization has a symmetry: $W_{b'b} = \overline{W_{-b,-b'}}$ *where the bar in the right-hand side means complex conjugation applying only on coefficients of the formal power series (i.e.* $\overline{\sum_k c_k \prod_e x_e^{k_e}} = \sum_k \overline{c_k} \prod_e x_e^{k_e}$*).*

3.2. Examples

Here are a few examples of plane graphs and the parametrizations (3) of the transition matrix W.

Example 1. *A plane graph with parallel edges is shown in figure 2(a). Let* $\varepsilon = \exp(\frac{\mathrm{i}}{2}(\frac{\pi}{2}))$. *The parametrization (3) for this graph is:* $W_{i,a}/x_a = \varepsilon^2, \varepsilon^3, \varepsilon^2$ *for* $i = d, -b, -c$ *respectively;* $W_{i,b}/x_b = \varepsilon, \varepsilon^{-1}, \varepsilon$ *for* $i = d, -a, -c$ *resp.;* $W_{i,c}/x_c = 1, \varepsilon^{-2}, \varepsilon^{-3}$ *for* $i = d, -a, -b$ *resp.;* $W_{e,d}/x_d = W_{f,e}/x_e = \varepsilon$*;* $W_{i,f}/x_f = 1, \varepsilon, \varepsilon^2$ *for* $i = a, b, c$ *resp.;* $W_{i,-a}/x_a = \varepsilon^{-3}, \varepsilon^{-2}, \varepsilon^{-2}$ *for* $i = b, c, -f$ *resp.;* $W_{i,-b}/x_b = \varepsilon, \varepsilon^{-1}, \varepsilon^{-1}$ *for* $i = a, c, -f$ *resp.;* $W_{i,-c}/x_c = \varepsilon^2, \varepsilon^3, 1$ *for* $i = a, b, -f$ *resp.;* $W_{i,-d}/x_d = 1, \varepsilon^{-1}, \varepsilon^{-2}$ *for* $i = -a, -b, -c$ *resp.;* $W_{-d,-e}/x_e = W_{-e,-f}/x_f = \varepsilon^{-1}$*; and* $W_{i,i'} = 0$ *for the other pairs* (i, i').

When $x_i = x$ *(constant) for all* $i \in E(G)$*, the right-hand side of (4) yields*

$$3x^2 - \tfrac{3}{2}x^4 + x^6 - \tfrac{3}{4}x^8 + \tfrac{3}{5}x^{10} - \tfrac{1}{2}x^{12} + \tfrac{3}{7}x^{14} - \tfrac{3}{8}x^{16} + \mathrm{O}(x^{17})$$

(here $\mathrm{O}(x^{17})$ *means that I calculated up to order* 16 *in x) which successfully reproduces the exact S:*

$$1 + 3x^2 + 3x^4 + x^6 + \mathrm{O}(x^{17}).$$

(The exact S is $1 + 3x^2 + 3x^4 + x^6$*, a polynomial of degree-six.)*

Example 2. *Figure 2(b) shows a plane graph with a closed edge. Again let* $\varepsilon = \exp(\frac{i}{2}(\frac{\pi}{2}))$. *The parametrization (3) becomes:* $W_{b,a}/x_a = \varepsilon$; $W_{i,b}/x_b = \varepsilon, \varepsilon^{-1}, 1$ *for* $i = c, e, -e$ *resp.;* $W_{d,c}/x_c = W_{a,d}/x_d = \varepsilon$; $W_{i,e}/x_e = \varepsilon^2, \varepsilon^4, \varepsilon^3$ *for* $i = c, e, -b$ *resp.;* $W_{-d,-a}/x_a = W_{-a,-b}/x_b = \varepsilon^{-1}$; $W_{i,-c}/x_c = 1, \varepsilon^{-1}, \varepsilon$ *for* $i = e, -b, -e$ *resp.;* $W_{-c,-d}/x_d = \varepsilon^{-1}$; $W_{i,-e}/x_e = \varepsilon^{-3}, \varepsilon^{-2}, \varepsilon^{-4}$ *for* $i = c, -b, -e$ *resp.; and* $W_{i,i'} = 0$ *otherwise.*

When $x_i = x$ *(constant) for all* $i \in E(G)$, *the right-hand side of (4) gives*

$$x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 + \tfrac{3}{4}x^4 + \tfrac{1}{5}x^5 - \tfrac{1}{6}x^6 + \tfrac{1}{7}x^7 - \tfrac{5}{8}x^8 + \tfrac{1}{9}x^9 - \tfrac{1}{10}x^{10} + \mathrm{O}(x^{11})$$

which yields the exact S*:*

$$1 + x + x^4 + x^5 + \mathrm{O}(x^{11}).$$

(The exact S *is* $1 + x + x^4 + x^5$*.)*

Example 3. *A graph embeddable into a cylinder is embeddable into a plane (and vice versa). So we can apply the formulae (4) and (5) for cylinder graphs.*

As an example let us consider the $n \times m$ *square lattice with cyclic boundary condition in the horizontal direction (i.e. the lattice is wrapped on a cylinder).*

One possible embedding of the graph into the plane is shown in figure 2(c) where only the edges $b_1, \ldots, b_m$ *('boundary edges') have nonzero curvatures, each of which gives* $\exp(\frac{i}{2}\int k\,\mathrm{d}s) = -1$ *in (3). Hence, in this case, we can restate the prescription (3) as follows: for allowed* $b' \to b$ $W_{bb'} = \widetilde{x_{b'}} \cdot \exp(\frac{i}{2}\varphi_{bb'})$ *where* $\widetilde{x_{b'}} = -x_{b'}$ *if* b' *goes across the boundary (i.e. if* $b' = b_i$ *for some* $i = 1, \ldots, m$ *in figure 2(c)) and* $\widetilde{x_{b'}} = x_{b'}$ *otherwise; in other words we can think that each edge is a straight segment (i.e. as if it has zero curvature) and/but the edges that go across the boundary acquire an extra factor* (-1)*.*

Of course, other embeddings yield different parametrizations. It is easily seen that the following gives another parametrization for the same $n \times m$ *square lattice with the cyclic boundary condition (see figure 2(d)):* $W_{bb'} = \widetilde{x_{b'}} \cdot \exp(\frac{i}{2}\varphi_{bb'})$ *where* $\widetilde{x_{b'}} = \exp(i\pi/n) \cdot x_{b'}$ *for right-directed horizontal edges* b', $\widetilde{x_{b'}} = \exp(-i\pi/n) \cdot x_{b'}$ *for left-directed horizontal* b', *and* $\widetilde{x_{b'}} = x_{b'}$ *for vertical* b'. *This parametrization has translation invariance in the horizontal direction.*

Let us give an explicit example of computations for the 3×3 *lattice with* $x_i = x$ *(constant). Both parametrizations give*

$$3x^3 + 6x^4 + 12x^5 + \tfrac{15}{2}x^6 - 21x^8 - 71x^9 - 132x^{10} - 132x^{11} + \cdots - \tfrac{166\,821}{7}x^{21} + \mathrm{O}(x^{22})$$

for $\log S$ *(4). It yields the exact* S*:*

$$1 + 3x^3 + 6x^4 + 12x^5 + 12x^6 + 18x^7 + 33x^8 + 28x^9 + 12x^{10} + 3x^{11} + \mathrm{O}(x^{22})$$

(the exact one is, of course, a polynomial of degree 11*).*

It seems that the observation of the first parametrization of this example leads to a recipe for torus graphs which will be described in the next section.

4. The partition function for a torus graph

This is the main section of this paper. We shall describe a new algorithm for counting polygons on a given torus graph which takes its topology (or boundary conditions) correctly into account.

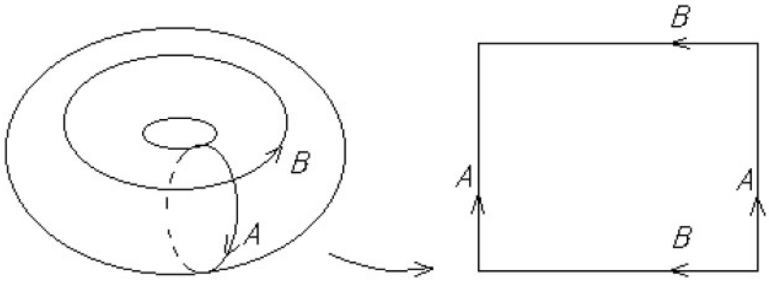

Figure 3. Dissecting the torus.

4.1. A random walk representation and a determinant formula

Let G be a finite graph which can be embedded into a torus (hence its genus is $\leqslant 1$). Let f be a smooth embedding of G into a torus. $M = \#E(G)$.

Again we want to consider a walk on the torus graph $f(G)$. In order to give a transitoin matrix W we consider a covering surface of the torus and the lift of the graph $f(G)$ drawn on the covering surface.

To do this let us dissect the torus in some standard way: that is, cut it open on two disjoint simple closed curves, A and B (they form a canonical basis of cycles on the torus), both beginning and ending at the same point, so that what remains is a rectangle or a 'tile' (see figure 3). We require that the two closed curves should be chosen so that (1) they do not pass through any vertex of $f(G)$; and (2) they intersect transversely with edges of $f(G)$. The torus graph $f(G)$ induces a figure on the rectangle tile. Prepare an infinite number of copies of the tile and let them cover the whole plane. Then we obtain a graph drawn on the plane which was induced from $f(G)$ on the torus; let us call the (infinite) graph drawn on the plane a *lift* $\widetilde{f(G)}$ of the torus graph $f(G)$.

An orientation on each edge $f(e)$, the set of all possible states $I = \{e|e \in E(G)\} \sqcup \{-e|e \in E(G)\}$ of walks, and the meaning of allowed transitions $b' \to b$ are defined in the same way as in the preceding section. Note that the orientation on $f(e)$ naturally induces an orientation on its lift. The transition matrix W for a walk on the torus graph $f(G)$ is defined as follows: for an allowed transition $b' \to b$ (where $b, b' \in I$)

$$W_{bb'} = \widetilde{x_{b'}} \cdot \exp\left(\frac{\mathrm{i}}{2}\left(\varphi_{bb'} + \int_{b'} k\, \mathrm{d}s\right)\right) \tag{6}$$

where $\varphi_{bb'}$ is the change in angle of the tangent vectors when a walk moves from b' to b at the terminal vertex of b' *measured on the lift*, k the geodesic curvature of the arc *on the lift* which corresponds to b', and s the arc length parameter for the lift corresponding to b'; the definition of $\widetilde{x_{b'}}$ is completely new: let us introduce two *commuting* variables a and b and define

$$\widetilde{x_{b'}} = a^m b^n x_e$$

if the edge b' transverses the closed curve A m times from left to right side (a transversing in the opposite direction is counted as -1), transverses the curve B n times from left to right, and $b' = e$ or $-e$. The variables a and b will play an essential role in our formalism. Of course, we define $W_{bb'} = 0$ if $b' \to b$ is not allowed.

Now we can describe the main formula.

Conjecture. *Let G be a finite graph embeddable into a torus and let f be a smooth embedding of G into a torus. Let W be a $2M \times 2M$ matrix defined by (6). (Note that W contains the two*

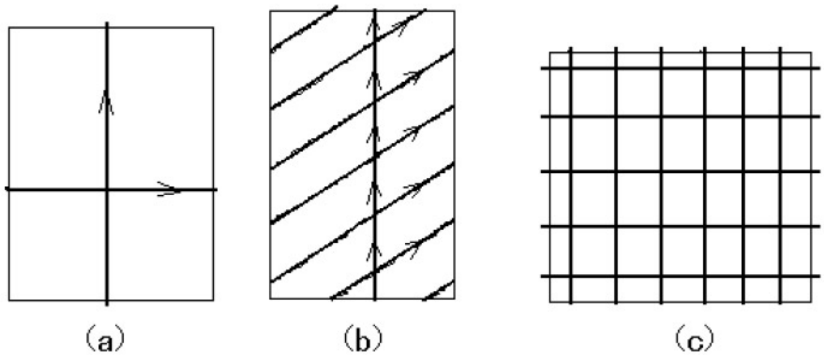

Figure 4. Torus graphs: (*a*) a bouquet; (*b*) the complete graph on five points; (*c*) the square lattice with cyclic boundary conditions in both directions. Each figure shows a figure drawn on a tile which was obtained by dissecting the torus.

commuting variables a and b.) Then the following formula holds for (2):

$$S(\{x_e | e \in E(G)\}) = \mathcal{H}\left(\exp\left(-\sum_{r=1}^{\infty} \frac{1}{2r} \operatorname{trace}(W^r) \right) \right). \tag{7}$$

The definition of the operation $\mathcal{H}$ is

$$\mathcal{H}\left(\sum_k c_k \prod_e x_e^{k_e} \right) = \sum_k \mathcal{H}(c_k) \prod_e x_e^{k_e}$$

(each c_k is a Laurent polynomial in a and b) and

$$\mathcal{H}\left(\sum c_{mn} a^m b^n \right) = \sum (-1)^{(m,n)} c_{mn}$$

[$c_{mn} \in \mathbb{C}$] where (m, n) denotes the greatest common divisor of the integers.

A heuristic explanation is as follows: the sum in the exponential in (7) expresses contributions from connected loops; hence the exponential itself is a certain weighted summation of polygons on $\widetilde{f(G)}$, the lift of $f(G)$ drawn *on the plane*; if we apply on it the operation $\mathcal{H}$ then all 'unwilling terms' which do not appear in (2) vanish and all coefficients of the remaining terms become correct ones. (Here (m, n) plays a role of 'winding number' of a curve on the torus.)

Unfortunately, we do not have a rigorous proof of this statement. However, for the simplest nontrivial torus graph, which is shown in figure 4(*a*), we shall see in the next section that formula (7) really holds; this observation bears out the conjecture.

As a corollary we obtain a determinant formula:

$$S(\{x_e | e \in E(G)\}) = \mathcal{H}([\det(I - W)]^{1/2}). \tag{8}$$

4.2. Examples

Let us show a few examples of torus graphs which we have used to check our formulae (7) and (8).

Example 1. *Consider a graph consisting of one vertex and two closed edges (such a graph is called a 'bouquet') and consider an embedding as shown in figure 4(a). This is the simplest nontrivial torus graph. Let us show explicit calculations to demonstrate how the formulae (7)* *and (8) work. Assign variables x, y to the horizontal and vertical edge, respectively, and let $\varepsilon = \exp(\frac{\mathrm{i}}{2}(\frac{\pi}{2}))$; the transition matrix W is*

$$\begin{array}{c} \rightarrow \\ \uparrow \\ \leftarrow \\ \downarrow \end{array} \begin{pmatrix} ax & \varepsilon^{-1}by & 0 & \varepsilon b^{-1}y \\ \varepsilon ax & by & \varepsilon^{-1}a^{-1}x & 0 \\ 0 & \varepsilon by & a^{-1}x & \varepsilon^{-1}b^{-1}y \\ \varepsilon^{-1}ax & 0 & \varepsilon a^{-1}x & b^{-1}y \end{pmatrix}$$

(arrows indicate indices). Then $-\sum \frac{1}{2r}\,\mathrm{trace}\,W^r$ in (7) is

$$\begin{aligned} &-\frac{1}{2}\left(\frac{1}{a}+a\right)x-\frac{1}{2}\left(\frac{1}{b}+b\right)y-\frac{1}{4}\left(\frac{1}{a^2}+a^2\right)x^2-\frac{1}{2}\left(\frac{1}{ab}+\frac{a}{b}+\frac{b}{a}+ab\right)xy\\ &\quad-\frac{1}{4}\left(\frac{1}{b^2}+b^2\right)y^2-\frac{1}{6}\left(\frac{1}{a^3}+a^3\right)x^3-\frac{1}{2}\left(\frac{1}{a^2b}+\frac{a^2}{b}+\frac{b}{a^2}+a^2b\right)x^2y\\ &\quad-\frac{1}{2}\left(\frac{1}{ab^2}+\frac{a}{b^2}+\frac{b^2}{a}+ab^2\right)xy^2-\frac{1}{6}\left(\frac{1}{b^3}+b^3\right)y^3-\frac{1}{8}\left(\frac{1}{a^4}+a^4\right)x^4\\ &\quad-\frac{1}{2}\left(\frac{1}{a^3b}+\frac{a^3}{b}+\frac{b}{a^3}+a^3b\right)x^3y\\ &\quad+\left(1-\frac{1}{2}\left(\frac{1}{a^2}+a^2\right)-\frac{1}{2}\left(\frac{1}{b^2}+b^2\right)-\frac{3}{4}\left(\frac{1}{a^2b^2}+\frac{a^2}{b^2}+\frac{b^2}{a^2}+a^2b^2\right)\right)x^2y^2\\ &\quad-\frac{1}{2}\left(\frac{1}{ab^3}+\frac{a}{b^3}+\frac{b^3}{a}+ab^3\right)xy^3-\frac{1}{8}\left(\frac{1}{b^4}+b^4\right)y^4+\cdots. \end{aligned}$$

This yields for $\mathrm{e}^{-\sum\frac{1}{2r}\mathrm{trace}\,W^r}$

$$\begin{aligned} &1-\frac{1}{2}\left(\frac{1}{a}+a\right)x-\frac{1}{2}\left(\frac{1}{b}+b\right)y+\left(\frac{1}{4}-\frac{1}{8}\left(\frac{1}{a^2}+a^2\right)\right)x^2\\ &\quad-\frac{1}{4}\left(\frac{1}{ab}+\frac{a}{b}+\frac{b}{a}+ab\right)xy\\ &\quad+\left(\frac{1}{4}-\frac{1}{8}\left(\frac{1}{b^2}+b^2\right)\right)y^2+\frac{1}{16}\left(-\frac{1}{a^3}-a^3+\frac{1}{a}+a\right)x^3\\ &\quad+\left(-\frac{3}{16}\left(\frac{1}{a^2b}+\frac{a^2}{b}+\frac{b}{a^2}+a^2b\right)+\frac{3}{8}\left(\frac{1}{b}+b\right)\right)x^2y\\ &\quad+\left(-\frac{3}{16}\left(\frac{1}{ab^2}+\frac{a}{b^2}+\frac{b^2}{a}+ab^2\right)+\frac{3}{8}\left(\frac{1}{a}+a\right)\right)xy^2\\ &\quad+\frac{1}{16}\left(-\frac{1}{b^3}-b^3+\frac{1}{b}+b\right)y^3\\ &\quad+\left(-\frac{5}{128}\left(\frac{1}{a^4}+a^4\right)+\frac{1}{32}\left(\frac{1}{a^2}+a^2\right)-\frac{1}{64}\right)x^4\\ &\quad+\left(-\frac{5}{32}\left(\frac{1}{a^3b}+\frac{a^3}{b}+\frac{b}{a^3}+a^3b\right)+\frac{5}{32}\left(\frac{1}{ab}+\frac{a}{b}+\frac{b}{a}+ab\right)\right)x^3y\\ &\quad+\bigg(-\frac{15}{64}\left(\frac{1}{a^2b^2}+\frac{a^2}{b^2}+\frac{b^2}{a^2}+a^2b^2\right)-\frac{1}{32}\left(\frac{1}{a^2}+a^2\right)\\ &\quad-\frac{1}{32}\left(\frac{1}{b^2}+b^2\right)+\frac{17}{16}\bigg)x^2y^2 \end{aligned}$$

$$+\left(-\frac{5}{32}\left(\frac{1}{ab^3}+\frac{a}{b^3}+\frac{b^3}{a}+ab^3\right)+\frac{5}{32}\left(\frac{1}{ab}+\frac{a}{b}+\frac{b}{a}+ab\right)\right)xy^3$$
$$+\left(-\frac{5}{128}\left(\frac{1}{b^4}+b^4\right)+\frac{1}{32}\left(\frac{1}{b^2}+b^2\right)-\frac{1}{64}\right)y^4+\cdots. \qquad (9)$$

(We computed up to eigth degree; to save space we have shown here results up to the fourth.) As for the determinant formula (8) we have

$$\det(I-W)=1-\left(\frac{1}{a}+a\right)x-\left(\frac{1}{b}+b\right)y+x^2+y^2+\left(\frac{1}{a}+a\right)xy^2+\left(\frac{1}{b}+b\right)x^2y+x^2y^2$$

which again yields (9) for $[\det(I-W)]^{1/2}$*. Applying on (9) the operation* $\mathcal{H}$ *we have*

$$1+x+y+xy$$

(up to eighth degree) which coincides with the exact S (2).

The computations on this example imply that formula (7) correctly takes account of the topology of torus graphs.

Example 2. *The complete graph on five points, denoted by* K_5*, is a graph of genus one. Consider an embedding of it into a torus as shown in figure 4(b). For this embedding formula (7) (and also (8)) gives*

$$S=1+10x^3+15x^4+12x^5+15x^6+10x^7+x^{10}$$

(we have computed up to order 20) which coincides with the exact one.

Example 3. *Consider the* $n\times m$ *square lattice with cyclic boundary conditions in both directions and an embedding into a torus as shown in figure 4(c).*

We have tested the formulae (7) and (8) for several small size lattices ($3\times 3, 3\times 4, 3\times 5, 4\times 4$*); they all gave exact results.*

Now consider the general $n\times m$ *lattice. Let* $\widetilde{W}$ *be a matrix given by (6) but with the definition of* $\widetilde{x_{b'}}$ *changed to*

$$\widetilde{x_{b'}}=\begin{cases} a^{1/n}x_e & \text{for horizontal right-directed } b' \\ a^{-1/n}x_e & \text{for horizontal left-directed } b' \\ b^{1/m}x_e & \text{for vertical up-directed } b' \\ b^{-1/m}x_e & \text{for vertical down-directed } b' \end{cases}$$

(where b' *corresponds to the edge* e*). Then, as is easily seen,* $\operatorname{trace}\widetilde{W}^r=\operatorname{trace}W^r$ *for any integer* r*; hence, we can use this* $\widetilde{W}$ *for our transition matrix. The use of* $\widetilde{W}$ *has an advantage because* $\widetilde{W}$ *has a (block-)translational invariance in both direction and hence can be (block-)diagonalized by a Fourier-like transform, just as was done in [9]. Hence formula (8) gives*

$$S=\left(\frac{\sinh 2K}{(\cosh K)^4}\right)^{\frac{1}{2}nm}\times\mathcal{H}\left(\left\{\prod_{p\in\mathbf{Z}_n,q\in\mathbf{Z}_m}\left(\frac{(\cosh 2K)^2}{\sinh 2K}\right.\right.\right.$$
$$\left.\left.\left.-\frac{1}{2}[a^{-\frac{1}{n}}\omega_n^p+a^{\frac{1}{n}}\omega_n^{-p}+b^{-\frac{1}{m}}\omega_m^q+b^{\frac{1}{m}}\omega_m^{-q}]\right)\right\}^{\frac{1}{2}}\right)$$

where $\omega_k=\exp(2\pi\mathrm{i}/k)$*.*

5. Concluding remarks

In section 3 we have described the most generalized version of Vdovichenko's formula for counting polygons on any *plane graph*.

In section 4 we have described a new algorithm for counting polygons on any *torus graph* which enables one to write down the Ising partition function for the torus graph. The principle is summarized as follows: in order to count connected loops we considered a random walk on the graph (following the original idea of Vdovichenko); in order to use a connected loop expansion we considered a covering surface of the torus, which was actually a plane, and a plane graph on it associated with the original torus graph; then a certain nonanalytic operation, which took account of the topology of the torus, yielded the correct partition function.

There should be a relation between Potts and Ward's method [4, 5] and ours. An investigation into this point, as well as a rigorous proof of the statement in section 4, are left for a further study.

References

[1] Kaufman B 1949 Crystal statistics II: partition function evaluated by spinor analysis *Phys. Rev.* **76** 1232–43

[2] Kac M and Ward J C 1952 A combinatorial solution of the two-dimensional Ising model *Phys. Rev.* **88** 1332–7

[3] Onsager L 1944 Crystal statistics I: a two-dimensional model with an order-disorder transition *Phys. Rev.* **65** 117–49

[4] Potts R B and Ward J C 1955 The combinatorial method and the two-dimensional Ising model *Prog. Theor. Phys.* **13** 38–46

[5] Kasteleyn P W 1961 The statistics of dimers on a lattice *Physica* **27** 1209–25

[6] Fisher M E 1961 Statistical mechanics of dimers on a plane lattice *Phys. Rev.* **124** 1664–72

[7] Hurst C A and Green H S 1960 New solution of the Ising problem for a rectangular lattice *J. Chem. Phys.* **33** 1059–62

[8] Kasteleyn P W 1963 Dimer statistics and phase transition *J. Math. Phys.* **4** 287–93

[9] Vdovichenko N V 1964 A calculation of the partition function for a plane dipole lattice *Zh. Eksp. Teor. Fys.* **47** 715–9

Vdovichenko N V 1965 *Sov. Phys.–JETP* **20** 477–9